Probability distribution of the conductance at the mobility edge

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Abstract

Distribution of the conductance $P(g)$ at the critical point of the metal-insulator transition is presented for three and four dimensional orthogonal systems. The form of the distribution is discussed. Dimension dependence of $P(g)$ is proven. The limiting cases $g \to \infty$ and $g \to 0$ are discussed in detail and relation $P(g) \to 0$ in the limit $g \to 0$ is proven

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As the conductance g in disordered systems is not the self-averaged quantity, the knowledge of its probability distribution is extremely important for our understanding of transport. This problem is of special importance at the critical point of the metal-insulator transition [1]. While the distribution of the conductance in the metallic phase is known to be Gaussian in agreement with the random-matrix theory [2] and the localized regime is characterized by the log-normal distribution of g [2], the form of the critical distribution remains still unknown. Among the problems which are not solved yet we mention e.g. the number of parameters which characterize distribution, the existence of huge fluctuations of the conductance, and the form of $P(g)$ for small values of g.

Several attempts has been made to characterize conductance distribution at the critical point. Using the Migdal - Kadanoff renormalization treatment, huge conductance fluctuations has been predicted in [3]. The same conclusion was found also in systems of dimension $d = 2 + \varepsilon$. In the limit $\varepsilon << 1$ the form of the distribution $P(g)$ was found analytically [4]. However, numerical studies of disordered 3D system [5] indicated that it is not possible to generalize these analytical conclusions for realistic 3D systems ($\varepsilon = 1$).

The form of $P(g)$ for 2D symplectic models was found in [6, 7]. Recently, $P(g)$ has been studied also for system in magnetic field, both in 3D [8] and in 2D [9]. The main conclusion of these studies is that the symmetry of the system influences the form of the distribution at the critical point more strongly than in the metallic or localized regime. Nevertheless, $P(g)$ is invariant with respect to the choice of the microscopic model within the same universality class [7].

Studies of the statistics of the conductance have their counterpart in the analysis of the level statistics $s = E_{i+1} - E_i$ of the eigenvalues of Hamiltonian [11]. The critical distribution $P(s)$ is also the subject of intensive studies within last years [12]. In particular, its dependence on the symmetry [13], and dimension [14] have been studied numerically.

In this Letter we present new numerical data for the 3D and 4D Anderson model (orthogonal ensembles). Although data prove the dimension dependence of the distribution, their enables us to discuss the common features of the critical distribution. In particular, we prove that $P(g)$ decreases more quickly than exponentially for large g. This assures that there are no huge fluctuations of the conductance, discussed in [3]. We prove also that $P(g) \to 0$ in the limit of $g \to 0$.

We calculated the conductance as

$$g = \mathrm{Tr}\ t^\dagger t = \sum \cosh^{-2}(z_i/2) \tag{1}$$

where quantities z_i determine eigenvalues of the transmission matrix $t^\dagger t$. Details of the method have been published elsewhere [5]. For a given system size L, the probability distribution of g has been calculated from an ensemble of $N_{\rm stat}$ samples. The list of used ensembles together with mean and variances of g are given in Table 1.

The last column of Table 1 presents parameter $\langle z_1\rangle$, which corresponds to the parameter Λ introduced in the finite size scaling theory by MacKinnon and Kramer [15] as $\langle z_1\rangle = \frac{2L_t}{L\Lambda}$ in the quasi-one dimensional limit $L^{d-1} \times L_t$, $L_t >> L$. When neglecting the smallest system size, our data confirm the L-invariance

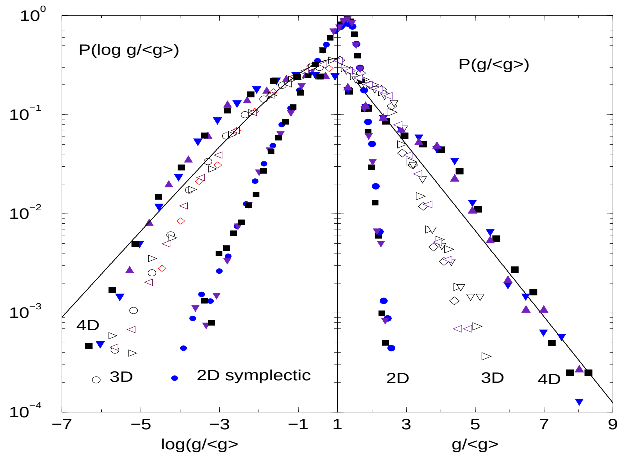

Figure 1: Probability distribution of $\log g/\langle g\rangle$ (left) and $g/\langle g\rangle$ (right) for 4D (full symbols) and 3D (open symbols) Anderson model. For comparison, we plot also data for 2D (symplectic) Ando model. The last exhibit the best convergence for both small and large values of g. For meaning of symbols, see Table 1. Solid line is Poisson distribution $P(g) = \exp -g/\langle g\rangle$.

of $\langle z_1\rangle$ as well as of $\langle g\rangle$ and $\langle \log g\rangle$ and their standard deviations. Owing to higher critical disorder, $\langle z_1\rangle$ is larger in 4D than in 3D. This guarantees that the finite-size effects disappear more quickly in 4D. Therefore, in spite of the fact that computer facilities limited the system size to $L \le 8$ for $d = 4$, obtained data provide us with the relevant information about all parameters of interest.

We presented in Table 1 both mean values of g and $\log g$ to underline the common features of 3D and 4D distribution: the variance of $\log g$ is of order of its mean value. This relation is typical for localized state. On the other hand, standard deviation of g is also $\sim \langle g\rangle$. Its value for 3D samples, 0.334, is smaller than the same quantity calculated for 3D in the metallic regime [5].

Numerical data for $P(g)$ are presented in Figure 1. They confirm that the critical distribution of g is system-size independent, in agreement with previous studies. Fig. 1. shows also that $P(g)$ depends on the dimension of the system within the same symmetry class. Although the distribution has the same form for 3D and 4D ensembles, it becomes broader for higher d: the probability to find $g << \langle g\rangle$ or $g >> \langle g\rangle$ growths with dimension. This is due to higher critical disorder, which causes that electronic state posses more features

Table 1: Review of ensembles studied in the present work. L-size of the d-dimensional cube, $N_{\rm stat}$: number of samples in a given ensemble, var $g = \langle g\rangle^2 - \langle g^2\rangle$, $\langle z_1\rangle$ is mean of the smallest of z's. Data for 3D AM are in good agreement with [9] (up to the spin degeneracy factor 2).

L		$N_{\rm stat}$	$\langle g\rangle$	$\sqrt{\rm var g}$	$\langle \log g\rangle$	var $\log g$	$\langle z_1\rangle$
3D Anderson model: $W_c \approx 16.5$							
6	$\circ$	20.000	0.375	0.324	-1.481	1.344	2.901
8	$\triangleleft$	20.000	0.400	0.333	-1.384	1.251	2.803
10	$\triangleright$	10.000	0.410	0.337	-1.347	1.229	2.770
12	$\diamond$	5.000	0.421	0.340	-1.302	1.199	2.724
14	$\triangle$	2.500	0.416	0.338	-1.306	1.122	2.725
18	$\bigtriangledown$	500	0.418	0.329	-1.279	1.083	2.717
4D Anderson model: $W_c \approx 34.5$ [12]							
4		22.000	0.190	0.247	-2.569	2.301	4.130
5	$\bigtriangledown$	30.000	0.229	0.270	-2.275	2.006	3.838
6	$\square$	15.000	0.225	0.269	-2.291	2.054	3.852
7	$\triangle$	7.000	0.239	0.275	-2.193	1.971	3.748
8		200	0.227	0.274	-2.188	1.692	3.790

of the localized state than that of the metallic one (remaining critical). This is in agreement with studies of the level statistics in 4D [12].

The small- g behavior of $P(g)$ can be estimated from Fig 1. Instead of $P(g/\langle g\rangle)$, we plot in the left side of Fig. 1. the distribution $\mathcal{P}(\gamma)$ of $\gamma = \log g/\langle g\rangle$. Evidently, $\log \mathcal{P}(\gamma) = \gamma + \log P(\exp\gamma)$. Therefore, an assumption $P(g = 0) = c \ne 0$, implies $\mathcal{P} = \gamma + \log c$ for $\gamma \to -\infty$.

Fig 1. shows clearly that $\log \mathcal{P}(\gamma)$ decreases more quickly than γ for *all* ensembles we consider. This guarantees that $P(g) \to 0$ as $g \to 0$. Let us note that it is almost impossible to obtain last result form the studies of $P(g)$ on the linear scale [10].

The small- g behavior of $P(g)$ is easy to estimate also from the distribution $P(z_1)$ of the smallest parameter z_1. Indeed, small values of g require large values of z_1. Neglecting contributions of other channels, we have

$$\frac{1}{2\varepsilon}\int_0^{2\varepsilon} P(g)dg = \frac{1}{2\varepsilon}\int_{\tilde{z_1}}^{\infty} P(z_1)dz_1 \qquad (2)$$

with $\varepsilon = \exp -\tilde{z_1}$. In the limit $\varepsilon \to 0$ the integral on the LHS reads $\sim P(g)$, $g = \varepsilon$. RHS could be found analytically for special form of $P(z_1)$. In particular, for Wigner surmises $P(z_1) = \pi/2\langle z_1\rangle^2 \times z_1 \exp(-\pi/4 \times [z_1/\langle z_1\rangle]^2)$ we obtain that $P(g) \sim g^{-1-{\rm const}\times \log g}/2$ with const $= \frac{\pi}{4\langle z_1\rangle^2}$. Consequently, $P(g = 0) = 0$. Fig. 2. assures that $P(z_1)$ decreases more quickly than

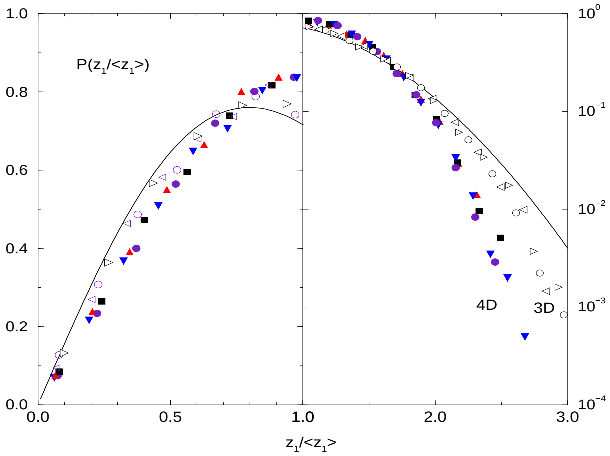

Figure 2: Probability distribution of (normalized) z_1 for 4D and 3D Anderson model. Solid line is Wigner surmises $P_W(z) = \frac{\pi}{2} z \exp[-\frac{\pi}{4} z^2]$. For the mean value $\langle z_1 \rangle$ see Table 1.

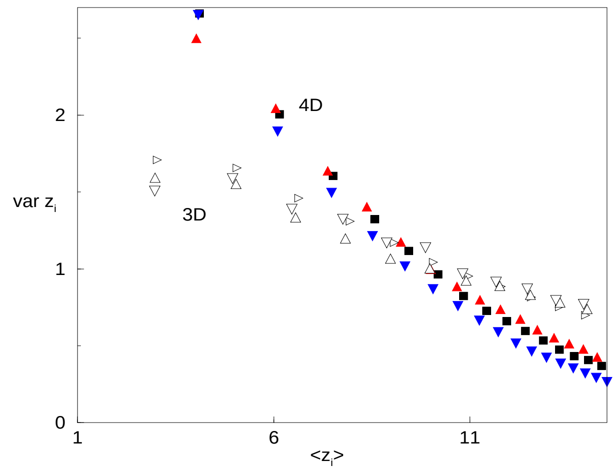

Figure 3: Var z_i as a function of $\langle z_i \rangle$ (only for $\langle z \rangle < 15$) for 3D (open symbols) and 4D (full symbols) orthogonal systems. Note the system-size invariance of presented parameters (at least for $i \leq L$).

Wigner surmise for large z_1 in orthogonal ensemble for both 3D and 4D systems. This assures that $P(g) \to 0$ as $g \to 0$.

Linear behavior of the distribution $P(z_1)$ for small z_1 (see left side of Fig. 2) guarantees nonzero probability that the first channel is fully open. Indeed, if $P(z_1) \sim C \times z_1$ for $z_1 \to 0$, then the probability, that the first channel contribution to the conductance, $g_1 = 1/\cosh^2(z_1)$, equals to 1, is C. This explains the origin of the characteristic bump in the distribution $P(g)$ for $g = 1$. In Fig. 1, the bump is clearly visible for both 3D and 4D systems.

Fig. 1 (right) confirms that $P(g)$ decreases more quickly than exponentially for large g. This is easy to understand on the basis of the analysis of the statistics of z's presented in [5]. Fig 3. shows mean values and variances of some smallest z's for both 3D and 4D system. Evidently, $\langle z_i \rangle \sim \mathcal{O}(1)$ and variances var z_i decreases quickly with index i. Consequently, the contribution to the conductance from the second (higher) channel is, due to (1), small (negligible). To estimate this contribution, we note that all higher z_i, $i \geq 2$, are normally distributed. [5]. Their mean and variances has been estimated as $\langle z_i \rangle \sim \langle z_1 \rangle \times i^{1/(d-1)}$ and var $z_i \sim \langle z_i \rangle^{-(d-2)}$ [16]. Although this result holds only in the quasi-one dimensional limit, where the mutual correlations of z's are negligible, they serve as a good quantitative estimation also for true d-dimensional cubes. As i is seen in Figure 3, this agreement is better for 4D than for 3D,. Then, the probability to find $g \approx n$ is less than $\exp[-\langle z_n \rangle / 2\text{var} z_n] \sim \exp[-\text{const} \times n^{d/(d-1)}]$ and

$$P(g) \sim \exp - \text{const} \times g^{d/(d-1)}. \tag{3}$$

We conclude that presented numerical data for 3D and 4D Anderson model prove the system size invariance of the conductance distribution at the critical point. Although the distribution depends on the dimension and symmetry of the system, we found its common features, namely exponential decrease of $P(g)$ for $g > 1$, and a decrease of $P(g)$ to zero for $g = 0$. We show that the form of $P(g)$ can be analyzed on the basis of the statistics of parameters z introduced by relation (1). This analysis is more simple for higher dimension, where the statistical correlations of zs are supposed to be less important.

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